Rearranging series of vectors on a small set

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Introduction

Theorem (Riemann)

For any conditionally convergent series of reals $\sum_{n=1}^{\infty} a_n$ *and any* $a \in \mathbb{R}$ *there exists a permutation* $\sigma : \mathbb{N} \to \mathbb{N}$ *such that* $\sum_{n=1}^{\infty} a_{\sigma(n)} = a$.

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Theorem (Wilczyński)

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$$\mathcal{I}_d = \{A \subseteq \mathbb{N} : \limsup_{n \to \infty} \frac{|A \cap \{0, 1, \ldots, n-1\}|}{n} = 0\}.$$

Definition

We say that an ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ *has the* (R) *property if for any conditionally convergent series of reals* $\sum_{n=1}^{\infty} a_n$ *and any* $a \in \mathbb{R}$ *there exists a permutation* $\sigma : \mathbb{N} \to \mathbb{N}$ *such that* $\sum_{n=1}^{\infty} a_{\sigma(n)} = a$ *and* $supp(\sigma) \in \mathcal{I}$*.*

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Theorem (Filipów, Szuca)

Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ *be an ideal. The following are equivalent.*

(i) $\mathcal{I}$ *has the* (R) *property.*

(ii) $\mathcal{I}$ *cannot be extended to a summable ideal.*

Definition

Let $(v_n)_n$ *be a sequence of vectors in* $\mathbb{R}^m$*.*
$S\left(\sum_{n=1}^{\infty} v_n\right) = \{v \in \mathbb{R}^m : \exists\sigma : \mathbb{N} \to \mathbb{N}$ *- permutation*
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The Lévy-Steinitz Theorem

Theorem (Lévy, Steinitz)

Let $(v_n)_n$ be a sequence of vectors in R^m. The set $S(\sum_{n=1}^{\infty} v_n)$ is either empty or is of the form $s_0 + L$ for some vector s_0 and some linear subspace L.

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Examples

$$S\left(\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n}, \frac{(-1)^n}{n}\right)\right) = \{(x, y) : x = y\},$$

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$$S\left(\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n}, \frac{(-1)^n}{\sqrt{n}}\right)\right) = \mathbb{R}^2.$$

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Also let s_0 be any sum (rearranged or not) of the series.
Finally, if $S(\sum_{n=1}^{\infty} v_n)$ is not empty, then

$$S\left(\sum_{n=1}^{\infty} v_n\right) = s_0 + F^\perp.$$

The two-dimensional case, ideal version

Theorem

Let $l \subseteq \mathbb{R}^2$ be such line on the plane and $\sum_{n=1}^{\infty} v_n$ such series in $\mathbb{R}^2$ that

$$S\left(\sum_{n=1}^{\infty} v_n\right) = l.$$

Then

$$S_{\mathcal{I}_d}\left(\sum_{n=1}^{\infty} v_n\right) = l.$$

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Alternatively, instead of $\mathcal{I}_d$ you can put any ideal that has the (R) property.

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Definition

We say that an ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ *has the* (R_2) *property if for any conditionally convergent series of vectors on the plane* $\sum_{n=1}^{\infty} v_n$ *such that* $S(\sum_{n=1}^{\infty} v_n) = \mathbb{R}^2$ *and any* $v \in \mathbb{R}^2$ *there exists a permutation* $\sigma : \mathbb{N} \to \mathbb{N}$ *such that* $\sum_{n=1}^{\infty} v_{\sigma(n)} = v$ *and* $supp(\sigma) \in \mathcal{I}$.

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Theorem (Folklore)

Let $(v_n)_n \subseteq \mathbb{R}^2$, $v_n \to 0$, $\forall w \neq 0 \ \sum_{n=1}^{\infty}(w \circ v_n)^+ = \infty$. Then

$$S\left(\sum_{n=1}^{\infty} v_n\right) = \mathbb{R}^2.$$

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Let $(v_n)_n \subseteq \mathbb{R}^2$, $v_n \to 0$. *The following are equivalent:*

- $S\left(\sum_{n=1}^{\infty} v_n\right) = \mathbb{R}^2$.
- *The set* $\left\{\sum_{n \in F} v_n : F \subseteq \mathbb{N}, |F| < \aleph_0\right\}$ *is dense in* $\mathbb{R}^2$.

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Theorem

Let $S\left(\sum_{n=1}^{\infty} v_n\right) = \mathbb{R}^2$. *There exists a set* $A \subseteq \mathbb{N}$ *such that both series* $\sum_{n\in A} v_n$ *and* $\sum_{n\in \mathbb{N}\setminus A} v_n$ *are conditionally convergent and* $S\left(\sum_{n\in A} v_n\right) = \mathbb{R}^2$, $S\left(\sum_{n\in \mathbb{N}\setminus A} v_n\right) = \mathbb{R}^2$.

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Corollary

If $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ *is a maximal ideal, then it has the* (R_2) *property.*

The two-dimensional case, ideal version

Remark

If $\sum_{n=1}^{\infty} a_n$ *is a series of reals such that* $a_n \to 0$, $\sum_{n=1}^{\infty} a_n^+ = \infty$ *and* $\sum_{n=1}^{\infty} a_n^- = -\infty$ *then there exists a subsequence* $(n_k)_k$ *such that* $\sum_{k=1}^{\infty} a_{n_k}$ *is conditionally convergent.*

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However, a similar situation is not true in the two-dimensional case.

The two-dimensional case, ideal version

Remark

If $\sum_{n=1}^{\infty} a_n$ is a series of reals such that $a_n \to 0$, $\sum_{n=1}^{\infty} a_n^+ = \infty$ and $\sum_{n=1}^{\infty} a_n^- = -\infty$ then there exists a subsequence $(n_k)_k$ such that $\sum_{k=1}^{\infty} a_{n_k}$ is conditionally convergent.

However, a similar situation is not true in the two-dimensional case.
That is, the following hypothesis

Hypothesis

If $(v_n)_n \subseteq \mathbb{R}^2$ is such that $v_n \to 0$ and for all $w \neq 0$ $\sum_{n=1}^{\infty}(w \circ v_n)^+ = \infty$, then there exists a subsequence $(n_k)_k$ such that $\sum_{k=1}^{\infty}(w \circ v_{n_k})^+ = \infty$ and the series $\sum_{k=1}^{\infty} v_{n_k}$ is convergent.

The two-dimensional case, ideal version

Remark

If $\sum_{n=1}^{\infty} a_n$ is a series of reals such that $a_n \to 0$, $\sum_{n=1}^{\infty} a_n^+ = \infty$ and $\sum_{n=1}^{\infty} a_n^- = -\infty$ then there exists a subsequence $(n_k)_k$ such that $\sum_{k=1}^{\infty} a_{n_k}$ is conditionally convergent.

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If $(v_n)_n \subseteq \mathbb{R}^2$ is such that $v_n \to 0$ and for all $w \neq 0$ $\sum_{n=1}^{\infty}(w \circ v_n)^+ = \infty$, then there exists a subsequence $(n_k)_k$ such that $\sum_{k=1}^{\infty}(w \circ v_{n_k})^+ = \infty$ and the series $\sum_{k=1}^{\infty} v_{n_k}$ is convergent.

is false.

The two-dimensional case, ideal version

Lemma

If $\sum_{n=1}^{\infty} x_n$ is a series of reals such that $x_n \to 0$, $\sum_{n=1}^{\infty} x_n^+ = \infty$ and $\sum_{n=1}^{\infty} x_n^- = -\infty$ then there exists $y_n \to 0$ such that for all $w \neq 0$ $\sum_{n=1}^{\infty}(w \circ (x_n, y_n))^+ = \infty$.

The two-dimensional case, ideal version

Lemma

If $\sum_{n=1}^{\infty} x_n$ *is a series of reals such that* $x_n \to 0$, $\sum_{n=1}^{\infty} x_n^+ = \infty$ *and* $\sum_{n=1}^{\infty} x_n^- = -\infty$ *then there exists* $y_n \to 0$ *such that for all* $w \neq 0$ $\sum_{n=1}^{\infty} (w \circ (x_n, y_n))^+ = \infty$.

Theorem

Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ *be an ideal. The following are equivalent.*

(i) *If* $(v_n)_n \subseteq \mathbb{R}^2$, $v_n \to 0$ *is such that* $\forall w \neq 0$ $\sum_{n=1}^{\infty} (w \circ v_n)^+ = \infty$ *then* $\exists A \in \mathcal{I}$ $\forall w \neq 0$ $\sum_{n \in A} (w \circ v_n)^+ = \infty$.

(ii) $\mathcal{I}$ *cannot be extended to a summable ideal.*

Thank you.